GPDs and SSAs

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Brief overview on generalized parton distributions (GPDs)

Probabilistic interpretation of GPDs as Fourier transforms of impact parameter dependent PDFs

\[ H(x, 0, -\Delta^2_\perp) \rightarrow q(x, b_\perp) \]

\[ \tilde{H}(x, 0, -\Delta^2_\perp) \rightarrow \Delta q(x, b_\perp) \]

\[ E(x, 0, -\Delta^2_\perp) \rightarrow \perp \text{ distortion of PDFs when the target is transversely polarized} \]

↩️ Sivers effect for single spin asymmetries

\[ 2\tilde{H}_T + E_T \rightarrow \perp \text{ distortion of } \perp \text{ polarized PDFs in unpolarized target} \]

↩️ correlation between quark angular momentum and quark spin

↩️ Boer-Mulders function \( h_1^\perp(x, k_\perp) \)

Summary
Generalized Parton Distributions (GPDs)

- GPDs: **decomposition of form factors** at a given value of \( t \), w.r.t. the average momentum fraction \( x = \frac{1}{2}(x_i + x_f) \) of the active quark

\[
\int dx H_q(x, \xi, t) = F_1^q(t) \quad \int dx \tilde{H}_q(x, \xi, t) = G_A^q(t)
\]

\[
\int dx E_q(x, \xi, t) = F_2^q(t) \quad \int dx \tilde{E}_q(x, \xi, t) = G_P^q(t),
\]

- \( x_i \) and \( x_f \) are the momentum fractions of the quark before and after the momentum transfer
- \( 2\xi = x_f - x_i \)

- **formal definition:**

\[
\int \frac{dx^-}{2\pi} e^{ix^-p^+x} \left\langle p' \left| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right\rangle = H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) + E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta^\nu}{2M} u(p)
\]
Generalized Parton Distributions (GPDs)

- measurement of the quark momentum fraction $x$ singles out one space direction (the direction of the momentum)
- makes a difference whether the momentum transfer is parallel, or $\perp$ to this momentum
- GPDs must depend on an additional variable which characterizes the direction of the momentum transfer relative to the momentum of the active quark $\rightarrow \xi$.
- in the limit of vanishing $t$ and $\xi$, the nucleon non-helicity-flip GPDs must reduce to the ordinary PDFs:

$$H_q(x, 0, 0) = q(x) \quad \tilde{H}_q(x, 0, 0) = \Delta q(x).$$

- GPDs are form factor for only those quarks in the nucleon carrying a certain fixed momentum fraction $x$
- $t$ dependence of GPDs for fixed $x$, provides information on the position space distribution of quarks carrying a certain momentum fraction $x$
### Form Factors vs. GPDs

<table>
<thead>
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<th>position space</th>
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<tbody>
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<td>$\bar{q}\gamma^+ q$</td>
<td>$Q$</td>
<td>$F(t)$</td>
<td>$\rho(\vec{r}^*)$</td>
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<tr>
<td>$\int \frac{dxc}{4\pi} e^{ix^+p^+x^-} \bar{q}\left(-\frac{x^-}{2}\right) \gamma^+ q\left(\frac{x^-}{2}\right)$</td>
<td>$q(x)$</td>
<td>$H(x, \xi, t)$</td>
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$$
\int \frac{dx^- e^{ix^+}}{4\pi} \bar{q}\left(\frac{-x^-}{2}\right) \gamma^+ q\left(\frac{x^-}{2}\right)
$$

$q(x, b_\perp) =$ impact parameter dependent PDF
Impact parameter dependent PDFs

- define state that is localized in \( \perp \) position:

\[
|p^+, \mathbf{R}_\perp = 0_\perp, \lambda\rangle \equiv \mathcal{N} \int d^2 \mathbf{p}_\perp |p^+, \mathbf{p}_\perp, \lambda\rangle
\]

Note: \( \perp \) boosts in IMF form Galilean subgroup \( \Rightarrow \) this state has

\[
\mathbf{R}_\perp \equiv \frac{1}{P^+} \int dx^- d^2 \mathbf{x}_\perp \mathbf{x}_\perp T^{++}(x) = \sum_i x_i \mathbf{r}_{i,\perp} = 0_\perp
\]

(cf.: working in CM frame in nonrel. physics)

- define impact parameter dependent PDF

\[
q(x, b_\perp) \equiv \int \frac{dx^-}{4\pi} \langle p^+, \mathbf{R}_\perp = 0_\perp | \bar{q}(-\frac{x^-}{2}, b_\perp) \gamma^+ q(\frac{x^-}{2}, b_\perp) | p^+, \mathbf{R}_\perp = 0_\perp \rangle e^{ixp^+ x^-}
\]
nucleon-helicity nonflip GPDs can be related to distribution of partons in \( \perp \) plane

\[
q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{i \Delta_\perp \cdot b_\perp} H(x, 0, -\Delta_\perp^2),
\]
\[
\Delta q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{i \Delta_\perp \cdot b_\perp} \tilde{H}(x, 0, -\Delta_\perp^2),
\]

- no rel. corrections to this result! (Galilean subgroup of \( \perp \) boosts)
- \( q(x, b_\perp) \) has probabilistic interpretation, e.g.

\[
q(x, b_\perp) \geq |\Delta q(x, b_\perp)| \geq 0 \quad \text{for} \quad x > 0
\]
\[
q(x, b_\perp) \leq |\Delta q(x, b_\perp)| \leq 0 \quad \text{for} \quad x < 0
\]

- Note that \( x \) already measures longitudinal momentum of quarks
- no simultaneous measurement of long. position of quarks
GPDs $\leftrightarrow q(x, b_{\perp})$

- $b_{\perp}$ distribution measured w.r.t. $R_{\perp}^{CM} \equiv \sum_i x_i r_{i,\perp}$
  - width of the $b_{\perp}$ distribution should go to zero as $x \to 1$, since the active quark becomes the $\perp$ center of momentum in that limit!
  - $H(x, 0, -\Delta_{\perp}^2)$ must become $\Delta_{\perp}^2$-indep. as $x \to 1$. Confirmed by recent [lattice studies](QCDSF, LHPC)

Anticipated shape of $q(x, b_{\perp})$:
- large $x$: quarks from localized valence ‘core’,
- small $x$: contributions from larger ‘meson cloud’
  - expect a gradual increase of the $t$-dependence (⊥ size) of $H(x, 0, t)$ as $x$ decreases
$q(x, b_\perp)$ in a simple model
Transversely Distorted Distributions and $E(x,0,-\Delta_{\perp}^2)$

- So far: only unpolarized (or long. pol.) nucleon! In general ($\xi = 0$):

\[
\int \frac{dx^-}{4\pi} e^{ip^+ x^- x} \langle P+\Delta, \uparrow | \bar{q}(0) \gamma^+ q(x^-) | P, \uparrow \rangle = H(x,0,-\Delta_{\perp}^2)
\]

\[
\int \frac{dx^-}{4\pi} e^{ip^+ x^- x} \langle P+\Delta, \uparrow | \bar{q}(0) \gamma^+ q(x^-) | P, \downarrow \rangle = -\frac{\Delta x - i\Delta y}{2M} E(x,0,-\Delta_{\perp}^2).
\]

- Consider nucleon polarized in $x$ direction (in IMF)

$|X\rangle \equiv |p^+, R_\perp = 0_\perp, \uparrow \rangle + |p^+, R_\perp = 0_\perp, \downarrow \rangle$.

$\rightarrow$ unpolarized quark distribution for this state:

\[
q(x,b_\perp) = \mathcal{H}(x,b_\perp) - \frac{1}{2M} \frac{\partial}{\partial b_y} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} E(x,0,-\Delta_{\perp}^2) e^{-i b_\perp \cdot \Delta_{\perp}}
\]

- Physics: $j^+ = j^0 + j^3$, and left-right asymmetry from $j^3$!
Electromagnetic interaction couples to vector current. Due to kinematics of the DIS-reaction (and the choice of coordinates — \( \hat{z} \)-axis in direction of the momentum transfer) the virtual photons “see” (in the Bj-limit) only the \( j^+ = j^0 + j^z \) component of the quark current.

If up-quarks have positive orbital angular momentum in the \( \hat{x} \)-direction, then \( j^z \) is positive on the \(+\hat{y}\) side, and negative on the \(-\hat{y}\) side.

\[
\begin{align*}
\vec{p}_\gamma & \quad \hat{y} \\
\hat{z} & \quad j^z > 0 \\
\hat{z} & \quad j^z < 0
\end{align*}
\]
Intuitive connection with $\vec{L}_q$

- Electromagnetic interaction couples to vector current. Due to kinematics of the DIS-reaction (and the choice of coordinates — $\hat{z}$-axis in direction of the momentum transfer) the virtual photons “see” (in the Bj-limit) only the $j^+ = j^0 + j^z$ component of the quark current.

- If up-quarks have positive orbital angular momentum in the $\hat{x}$-direction, then $j^z$ is positive on the $+\hat{y}$ side, and negative on the $-\hat{y}$ side.

- $j^+$ is distorted not because there are more quarks on one side than on the other but because the DIS-photons (coupling only to $j^+$) “see” the quarks on the $+\hat{y}$ side better than on the $-\hat{y}$ side (for $L_x > 0$).
Transversely Distorted Distributions and $E(x, 0, -\Delta^2_\perp)$

- $q(x, b_\perp)$ in $\perp$ polarized nucleon is distorted compared to longitudinally polarized nucleons!

- Mean $\perp$ displacement of flavor $q$ ($\perp$ flavor dipole moment)

$$d_y^q \equiv \int dx \int d^2b_\perp q_X(x, b_\perp)b_y = \frac{1}{2M} \int dx E_q(x, 0, 0) = \frac{\kappa_q^p}{2M}$$

with $\kappa_{u/d}^p \equiv F_{2u/d}^p(0) = O(1-2) \Rightarrow d_y^q = O(0.2\text{ fm})$

- Simple model: for simplicity, make ansatz where $E_q \propto H_q$

$$E_u(x, 0, -\Delta^2_\perp) = \frac{\kappa_{u}^p}{2} H_u(x, 0, -\Delta^2_\perp)$$

$$E_d(x, 0, -\Delta^2_\perp) = \kappa_{d}^p H_d(x, 0, -\Delta^2_\perp)$$

with $\kappa_{u}^p = 2\kappa_p + \kappa_n = 1.673$ \hspace{1cm} $\kappa_{d}^p = 2\kappa_n + \kappa_p = -2.033$.

- Model too simple but illustrates that anticipated distortion is very significant since $\kappa_u$ and $\kappa_d$ known to be large!
SSA \((\gamma + p \uparrow \rightarrow \pi^+ + X)\)

- Use factorization (high energies) to express momentum distribution of outgoing \(\pi^+\) as convolution of:
  - Momentum distribution of quarks in nucleon
  - Unintegrated parton density \(q(x, k_{\perp})\)
  - Momentum distribution of \(\pi^+\) in jet created by leading quark \(q\)
  - Fragmentation function \(D^\pi_\pi(z, p_{\perp})\)

- Average \(\perp\) momentum of pions obtained as sum of:
  - Average \(k_{\perp}\) of quarks in nucleon (Sivers effect)
  - Average \(p_{\perp}\) of pions in quark-jet (Collins effect)
**Sivers distribution**

- **Sivers**: Momentum distribution of **unpolarized** quarks in a \( \perp \) polarized proton

\[
f_{q/p^\uparrow}(x, k_\perp) = f_1^q(x, k_\perp^2) - f_{1T}^q(x, k_\perp^2) \left( \hat{P} \times k_\perp \right) \cdot S \]

\( \hat{P} \) is the polarization vector of the proton.
without FSI, $\langle k_\perp \rangle = 0$, i.e. $f_{1T}^q(x, k_\perp^2) = 0$

Brodsky, Hwang, Schmidt: simple model, which demonstrated that, including FSI, $\langle k_\perp \rangle \neq 0$

FSI formally included by appropriate choice of Wilson line gauge links (Boer et al; Collins; Ji et al;...) in gauge invariant def. of $q(x, k_\perp)$

$\leftrightarrow$ Qiu, Sterman; Boer et al.;..

\[
\langle k_\perp \rangle \sim \left\langle P, S \left\vert \bar{q}(0)\gamma^+ \int_0^\infty d\eta^- G^{+\perp}(\eta)q(0) \right\vert P, S \right\rangle
\]

$\int_0^\infty d\eta^- G^{+\perp}(\eta)$ is the $\perp$ impulse that the active quark acquires as it moves through color field of “spectators”

What should we expect for Sivers effect in QCD?

What do we learn if we measure $k_\perp$-asymmetry?
example: $\gamma p \rightarrow \pi X$ (Breit frame)

$u, d$ distributions in $\perp$ polarized proton have left-right asymmetry in $\perp$ position space ($T$-even!); sign determined by $\kappa_u$ & $\kappa_d$

attractive FSI deflects active quark towards the center of momentum
treat FSI to lowest order in $g$

$$\langle k^i_q \rangle = -\frac{g}{4p^+} \int \frac{d^2b_\perp}{2\pi} \frac{b^i}{|b_\perp|^2} \left\langle p, s \ \left| \bar{q}(0) \gamma^+ \frac{\lambda_a}{2} q(0) \rho_a(b_\perp) \right| p, s \right\rangle$$

with $\rho_a(b_\perp) = \int dr^- \rho_a(r^-, b_\perp)$ summed over all quarks and gluons

SSA related to dipole moment of density-density correlations

GPDs (N polarized in $+\hat{x}$ direction): $u \rightarrow +\hat{y}$ and $d \rightarrow -\hat{y}$

expect density density correlation to show same asymmetry

$$\langle b^y \bar{u}(0) \gamma^+ \frac{\lambda_a}{2} u(0) \rho_a(b_\perp) \rangle > 0$$

sign of SSA opposite to sign of distortion in position space
GPD $\leftrightarrow$ SSA (Sivers)

- example: $\gamma p \rightarrow \pi X$ (Breit frame)

- $u, d$ distributions in $\perp$ polarized proton have left-right asymmetry in $\perp$ position space (T-even!); sign determined by $\kappa_u$ & $\kappa_d$

- attractive FSI deflects active quark towards the center of momentum

- FSI translates position space distortion (before the quark is knocked out) in $+\hat{y}$-direction into momentum asymmetry that favors $-\hat{y}$ direction

- correlation between sign of $\kappa_q$ and sign of SSA: $f_{1T}^{\perp q} \sim -\kappa_q$

- signs of SSA confirmed at HERMES
Chirally Odd GPDs

\[ \int \frac{dx^-}{2\pi} e^{ip^+x^-} \left\langle p' \left| q \left( -\frac{x^-}{2} \right) \sigma^+ j_5 q \left( \frac{x^-}{2} \right) \right| p \right\rangle = H_T \bar{u} \sigma^+ j_5 u + \tilde{H}_T \bar{u} \frac{\epsilon^+ j_\alpha \beta \Delta_\alpha P_\beta}{M^2} u + E_T \bar{u} \frac{\epsilon^+ j_\alpha \beta \Delta_\alpha \gamma_\beta}{2M} u + \tilde{E}_T \bar{u} \frac{\epsilon^+ j_\alpha \beta P_\alpha \gamma_\beta}{M} u \]

- See also M. Diehl + P. Hägler, hep-ph/0504175.

- Fourier trafo of \(2\tilde{H}_T^q + E_T^q\), for \(\xi = 0\) describes distribution of transversity for unpolarized target in \(\perp\) plane

\[ q^i(x, b_\perp) = \frac{\epsilon^{ij}}{2M} \frac{\partial}{\partial b_j} \int \frac{d^2 \Delta_\perp}{(2\pi)^2} e^{ib_\perp \cdot \Delta_\perp} \left[ 2\tilde{H}_T^q(x, 0, -\Delta_\perp^2) + E_T^q(x, 0, -\Delta_\perp^2) \right] \]

- origin: correlation between quark spin (i.e. transversity) and angular momentum
Transversity Distribution in Unpolarized Target
Transversity Distribution in Unpolarized Target

- attractive FSI expected to convert position space asymmetry into momentum space asymmetry

\[ \rightarrow \] e.g. quarks at negative \( b_x \) with spin in \( +\hat{y} \) get deflected (due to FSI) into \( +\hat{x} \) direction

\[ \rightarrow \] (qualitative) connection between Boer-Mulders function \( h^\perp_{1T}(x, k_\perp) \) and the chirally odd GPD \( 2\tilde{H}_T + E_T \) that is similar to (qualitative) connection between Sivers function \( f^\perp_{1T}(x, k_\perp) \) and the GPD \( E \).
Sivers vs. Boer-Mulders distribution

- **Sivers**: Momentum distribution of *unpolarized* quarks in a ⊥ polarized proton

\[ f_{q/p}^{\uparrow}(x, k_\perp) = f_1^q(x, k_\perp^2) - f_{1T}^q(x, k_\perp^2) \frac{(\hat{P} \times k_\perp) \cdot S}{M} \]

- **Boer-Mulders**: Momentum distribution of ⊥ polarized quarks in an unpolarized proton

\[ f_{q/p}^{\uparrow}(x, k_\perp) = f_1^q(x, k_\perp^2) - h_{1T}^q(x, k_\perp^2) \frac{(\hat{P} \times k_\perp) \cdot S_q}{M} \]
Transversity Distribution in Unpolarized Target

- attractive FSI expected to convert position space asymmetry into momentum space asymmetry
- e.g. quarks at negative $b_x$ with spin in $+\hat{y}$ get deflected (due to FSI) into $+\hat{x}$ direction
- (qualitative) connection between Boer-Mulders function $h_{1T}^+(x, k_\perp)$ and the chirally odd GPD $2\tilde{H}_T + E_T$ that is similar to (qualitative) connection between Sivers function $f_{1T}^+(x, k_\perp)$ and the GPD $E$.
- qualitative predictions for $h_{1T}^+(x, k_\perp)$
  - sign of $h_{1T}^+$ opposite sign of $2\tilde{H}_T + E_T$
  - $\frac{h_{1T}^+}{2\tilde{H}_T + E_T} \approx \frac{f_{1T}^+}{E}$``
- use measurement of $h_{1T}^+$ to learn about spin-orbit correlation in nucleon wave function
- use LGT calcs. of $2\tilde{H}_T + E_T$ to make qualitative prediction for $h_{1T}^+$
GPDs provide decomposition of form factors w.r.t. the momentum of the active quark

\[
\int \frac{dx^-}{2\pi} e^{i p^+ x^-} \left\langle p' \left| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right\rangle
\]

GPDs resemble both PDFs and form factors: defined through matrix elements of light-cone correlator, but \( \Delta \equiv p' - p \neq 0 \).

t-dependence of GPDs at \( \xi = 0 \) (purely \( \perp \) momentum transfer) \( \Rightarrow \) Fourier transform of impact parameter dependent PDFs \( q(x, b_\perp) \)

\( \leftrightarrow \) knowledge of GPDs for \( \xi = 0 \) provides novel information about nonperturbative parton structure of nucleons: distribution of partons in \( \perp \) plane

\[
q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta_\perp^2) e^{i b_\perp \cdot \Delta_\perp}
\]

\[
\Delta q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} \tilde{H}(x, 0, -\Delta_\perp^2) e^{i b_\perp \cdot \Delta_\perp}
\]

\( q(x, b_\perp) \) has probabilistic interpretation, e.g. \( q(x, b_\perp) > 0 \) for \( x > 0 \)
\Delta_{2M}^T E(x, 0, -\Delta_{\perp}^2) \) describes how the momentum distribution of unpolarized partons in the \( \perp \) plane gets transversely distorted when is nucleon polarized in \( \perp \) direction.

(attractive) final state interaction in semi-inclusive DIS converts \( \perp \) position space asymmetry into \( \perp \) momentum space asymmetry

\[ 2\tilde{H}_T + E_T \] measures correlation between \( \perp \) spin and \( \perp \) angular momentum

physical explanation for Boer-Mulders effect; suggested relation between \( h_{\perp}^+ \) and the GPDs \( 2\tilde{H}_T + E_T \)


Connection to SSA in M.B., PRD 69, 057501 (2004); NPA 735, 185 (2004); PRD 66, 114005 (2002).